The large-time asymptotics for the modified Camassa-Holm equation on a non-zero background

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Cauchy problem for the mCH equation

The modified Camassa–Holm (mCH) equation:

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m = u - u_{xx}.$$

Cauchy problem:

$$u(x,0) = u_0(x), \qquad -\infty < x < \infty,$$

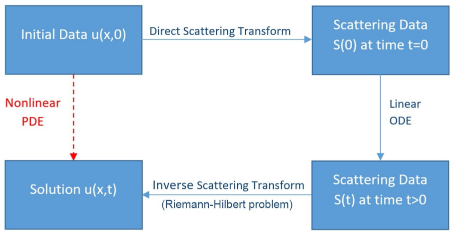
where $u_0(x) \to 1$ as $|x| \to \infty$.

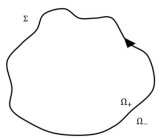
- Introducing $\tilde{u}$ such that $u(x,t) = \tilde{u}(x-t,t) + 1$, the mCH equation becomes

$$\tilde{m}_t + \left((\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}\right)_x = 0, \qquad \tilde{m} = \tilde{u} - \tilde{u}_{xx} + 1$$

on the zero background ($\tilde{u} \to 0$ as $|x| \to \infty$).

Approach: Riemann–Hilbert problem





Riemann–Hilbert problem: boundary value problem in complex analysis

Given contour $\Sigma \in \overline{\mathbb{C}}$ and "jump function" $G(s) : \Sigma \mapsto \mathbb{C}^{n\times n}$, find a function $M(z) : \mathbb{C} \setminus \Sigma \mapsto \mathbb{C}^{n\times n}$ such that:

- $M(z)$ is analytic in $\mathbb{C} \setminus \Sigma$;
- $M_+(s) = M_-(s)G(s),\ s \in \Sigma$;
- $M(\infty) = I$.

- In applications, the jump matrix G depends on parameters; then the solution also depends on these parameters; e.g., $G(s; x, t) \mapsto M(z; x, t)$

Inverse scattering transform method in the RHP form

- using
  - the Lax pair associated to the mCH equation
- construct
  - a multiplicative matrix Riemann-Hilbert problem (RHP)
- obtain
  - a representation of the solution $u(x,t)$ of the mCH equation in terms of the solution $M(x,t;\cdot)$ of the associated RHP
- obtain
  - the long-time asymptotics of $u(x,t)$ via the Deift-Zhou nonlinear steepest descent method.

mCH as integrable equation

- This equation is the compatibility condition for the Lax pair equations:

$$\begin{cases} \Phi_x(x,t,\lambda) = U(x,t,\lambda)\Phi(x,t,\lambda) \\ \Phi_t(x,t,\lambda) = V(x,t,\lambda)\Phi(x,t,\lambda) \end{cases},$$

where U and V are 2×2 matrices depending on the spectral parameter λ: $U = \frac{1}{2}\begin{pmatrix} -1 & \lambda\tilde{m} \\ -\lambda\tilde{m} & 1 \end{pmatrix}$;

$$V = \begin{pmatrix} \lambda^{-2} + \frac{(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})}{2} & -\lambda^{-1}(\tilde{u} - \tilde{u}_x + 1) - \frac{\lambda(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}}{2} \\ \lambda^{-1}(\tilde{u} + \tilde{u}_x + 1) + \frac{\lambda(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}}{2} & -\lambda^{-2} - \frac{(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})}{2} \end{pmatrix}.$$

- In order to control large λ behaviour, we introduce a new spatial variable

$$y(x,t) = x - \int_x^{+\infty} (\tilde{m}(\xi,t) - 1)\,d\xi.$$

Obtaining $\tilde{u}(x,t)$

Consider the solitonless case assuming that there are no residue conditions.

1. Given $u_0(x)$, construct the "reflection coefficient" $r(\mu)$, $\mu \in \mathbb{R}$, by solving the Lax pair equations, whose coefficients are determined in terms of $u_0(x)$.
2. Construct the jump matrix $J(y,t,\mu)$, $\mu \in \mathbb{R}$ by

$$J(y,t,\mu) = e^{-p(y,t,\mu)\sigma_3} J_0(\mu) e^{p(y,t,\mu)\sigma_3}$$

(J explicitly depends on y and t) where

$$p(y,t,\mu) = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y + \frac{8\mu^2}{(\mu^2+1)^2}t\right)$$

and $J_0(\mu)$ is defined by

$$J_0(\mu) = \begin{pmatrix} 1 - r(\mu)r^*(\mu) & r(\mu) \\ -r^*(\mu) & 1 \end{pmatrix}.$$

3. Solve the following RH problem: Find a piece-wise meromorphic, 2×2-matrix valued function $M(y,t,\mu)$ satisfying:
   - $M_+(y,t,\mu) = M_-(y,t,\mu)\mathrm{e}^{-p\sigma_3}\begin{pmatrix} 1 - r(\mu)r^*(\mu) & r(\mu) \\ -r^*(\mu) & 1 \end{pmatrix}\mathrm{e}^{p\sigma_3}$, $\mu \in \mathbb{R} \setminus \{-1, 1\}$, where $p := p(y,t,\mu) = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y + \frac{8\mu^2}{(\mu^2+1)^2}t\right)$ and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
   - $M \to I$ as $\mu \to \infty$
   - $M_\pm(y,t,\mu) = \frac{\mathrm{i}}{2(\mu-1)}\alpha_\pm(y,t)\left(\begin{smallmatrix} -c & 1 \\ -c & 1 \end{smallmatrix}\right) + O(1)$ as $\mu \to 1$
   - $M_\pm(y,t,\mu) = \frac{\mathrm{i}}{2(\mu+1)}\alpha_\pm(y,t)\left(\begin{smallmatrix} -c & -1 \\ c & 1 \end{smallmatrix}\right) + O(1)$ as $\mu \to -1$
   - $M(\cdot,\cdot,\frac{1}{\mu}) = \overline{M}(\cdot,\cdot,\overline{\mu}) = \sigma_3 M(\cdot,\cdot,-\mu)\sigma_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)M(\cdot,\cdot,\mu)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$,
4. Expand $M(y,t,\mu)$ in the neighborhood of i in order to obtain $a(y,t)$, $a_2(y,t)$, $a_3(y,t)$:

$$M(y,t,\mu) = \begin{pmatrix} a(y,t) & 0 \\ 0 & \frac{1}{a(y,t)} \end{pmatrix} + \begin{pmatrix} 0 & a_2(y,t) \\ a_3(y,t) & 0 \end{pmatrix}(\mu - \mathrm{i}) + O((\mu-\mathrm{i})^2)$$

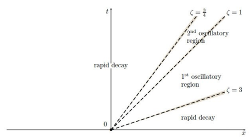
Then $\tilde{u}$ is given in a parametric form:

- $\tilde{u}(y,t) = -\frac{a_3(y,t)}{a(y,t)} - a_2(y,t)a(y,t)$
- $\tilde{u}_x(y,t) = \frac{a_3(y,t)}{a(y,t)} - a_2(y,t)a(y,t)$
- $x(y,t) = y + 2\ln a(y,t)$

Asymptotics

Question

How does $u(x,t)$ behave for large t?

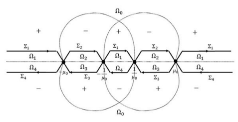


Answer

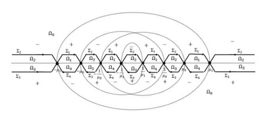
- four sectors in the (x,t) half-plane where $u(x,t)$ behaves differently for large t, depending on the magnitude of $\zeta = \frac{x}{t}$

Signature table:

Sign of $\mathrm{Im}\,\frac{p(y,t,\mu)}{\mathrm{i}t} = \frac{1}{4}(\mu - \frac{1}{\mu})(\xi - \frac{2}{(\mu+\frac{1}{\mu})^2})$ for various $\xi = \frac{y}{t}$; μ near $\mathbb{R}$



$1 < \zeta < 3$



$\frac{3}{4} < \zeta < 1$

$1 < \frac{x}{t} < 3$

In the solitonless case, the solution $u(x,t)$ of the Cauchy problem for the mCH equation has the following large-time asymptotics in the sector of the (x,t) half-plane defined by $1 < \zeta := \frac{x}{t} < 3$:

$$u(x,t) = 1 + \frac{C_1(\zeta-1)}{\sqrt{t}}\cos\left\{C_2(\zeta-1)t + C_3(\zeta-1)\ln t + \tilde{C}_4(\zeta-1)\right\} + o(t^{-1/2}),$$

where C_i are functions of ζ specified in terms of the scattering data, which in turn are uniquely specified by the initial data.
The error term is uniform in any sector $1+\varepsilon < \zeta < 3-\varepsilon$ where ε is a small positive number.

$\frac{3}{4} < \frac{x}{t} < 1$

In the solitonless case, the solution $u(x,t)$ of the Cauchy problem for the mCH equation has the following large-time asymptotics along the rays $\frac{x}{t} := \zeta$ in the sector of the (x,t) half-plane defined by $\frac{3}{4} < \zeta < 1$:

$$u(x,t) = 1 + \sum_{j=0,1}\frac{C_1^{(j)}(\zeta-1)}{\sqrt{t}}\cos\left(C_2^{(j)}(\zeta-1)t + C_3^{(j)}(\zeta-1)\ln t + \tilde{C}_4^{(j)}(\zeta-1)\right) + o(t^{-1/2}),$$

where $C_i^{(j)}$ are functions of ζ specified in terms of the scattering data, which in turn are uniquely specified by the initial data.
The error term is uniform in any sector $\frac{3}{4}+\varepsilon < \zeta < 1-\varepsilon$ where ε is small and positive.

References

[1] Anne Boutet de Monvel, Iryna Karpenko, Dmitry Shepelsky, The modified Camassa-Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem, preprint: arXiv:2011.13235
[2] Anne Boutet de Monvel, Iryna Karpenko, Dmitry Shepelsky, A Riemann-Hilbert approach to the modified Camassa-Holm equation with nonzero boundary conditions, Journal of Mathematical Physics, Volume 61, Issue 3 (2020), 031504.